

# RELATIONS BETWEEN A TOPOLOGICAL GAME AND THE $G_\delta$ -DIAGONAL PROPERTY

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**ABSTRACT.** We present a selection principle  $S_1(\mathcal{O}, \mathcal{H})$  that characterizes the  $G_\delta$ -diagonal property. We also present a topological game induced by this selection principle and we study the relations between this game and the  $G_\delta$ -property. Finally, we give some applications and examples.

## 1. INTRODUCTION

Let  $X$  be a topological space and let  $\mathcal{O}$  be the set of all open covers for a space  $X$ . Given  $\mathcal{C} \in \mathcal{O}$  define  $St(x, \mathcal{C}) = \bigcup \{C \in \mathcal{C} : x \in C\}$ .

The **diagonal** of the  $X \times X$  is the subset  $\Delta = \{(x, x) : x \in X\}$ . We say that  $X$  has the  $G_\delta$ -**diagonal property** if  $\Delta$  is a  $G_\delta$  subset of  $X \times X$ .

We say that  $\mathcal{A} \in \mathcal{O}$  is a **point-finite cover** if, for every  $x \in X$ , the set  $\{A \in \mathcal{A} : x \in A\}$  is finite. We say that a space  $(X, \tau)$  is a **metacompact space** if every open cover has an open refinement that is point-finite.

Let  $X$  be a topological space and consider  $L(X) = \min\{\kappa \in \omega : \text{given } \mathcal{C} \in \mathcal{O} \text{ there is a } \mathcal{C}' \subset \mathcal{C} \text{ such that } \bigcup \mathcal{C}' = X \text{ and } |\mathcal{C}'| \leq \kappa\}$ . We call this ordinal  $L(X)$  the **Lindelöf degree** of the space  $X$ .

Along this work we will use the standard topological definitions, following [3].

## 2. RELATIONS BETWEEN $S_1(\mathcal{O}, \mathcal{H})$ AND $G_1(\mathcal{O}, \mathcal{H})$

Recall the following characterization for the  $G_\delta$ -diagonal property.

**Theorem 2.1** (Ceder,[2]). *Let  $(X, \tau)$  be a topological space. Then  $X$  has the  $G_\delta$ -diagonal property if, and only if, there is a countable sequence of open covers  $(\mathcal{C}_n)_{n \in \omega} \subset \mathcal{O}$  such that, for every  $x, y \in X$ , with  $x \neq y$ , there is a  $k \in \omega$  such that  $y \notin St(x, \mathcal{C}_k)$ . In other words, for each  $x \in X$ ,  $\bigcap_{n \in \omega} St(x, \mathcal{C}_n) = \{x\}$ .*

This  $G_\delta$ -diagonal characterization give us motivation for a selection principle.

Let  $(X, \tau)$  be a topological space. Let  $\mathcal{H} = \{R \in (\tau \setminus \{\emptyset\})^\omega : |\bigcap R| \geq 2\}$ . The notation  $S_1(\mathcal{O}, \mathcal{H})$  abbreviates the following statement:

Given  $(\mathcal{C}_n)_{n \in \omega} \subset \mathcal{O}$ , for each  $n \in \omega$  there is a  $C_n \in \mathcal{C}_n$  such that  $|\bigcap_{n \in \omega} C_n| \geq 2$ .

Associated to this principle, there is a game  $G_1(\mathcal{O}, \mathcal{H})$  defined as follows.

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In the  $n$ -th inning, Player I plays a  $\mathcal{C}_n \in \mathcal{O}$  and Player II chooses a  $C_n \in \mathcal{C}_n$ . At the end, Player II is the winner if  $|\bigcap_{n \in \omega} C_n| \geq 2$ .

Note that:

$$\Delta \text{ is } G_\delta \text{ in } X^2 \Leftrightarrow \neg S_1(\mathcal{O}, \mathcal{H}) \Rightarrow \text{I} \uparrow G_1(\mathcal{O}, \mathcal{H}).$$

Where  $\text{I} \uparrow G_1(\mathcal{O}, \mathcal{H})$  means that Player I has a winning strategy for  $G_1(\mathcal{O}, \mathcal{H})$ .

In the following, we will discuss when the second implication can be reversed.

**Proposition 2.2.** *Let  $X$  be a Lindelöf space. If Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$  then  $S_1(\mathcal{O}, \mathcal{H})$  does not hold.*

*Proof.* Without loss of generality, we can assume that at each inning Player I plays a countable open covering. Then a winning strategy for Player I can be identified with a family  $\{\mathcal{C}_\rho : \rho \in \omega^{<\omega}\}$  such that  $\mathcal{C}_\rho = \{A_{\rho \frown n} : n \in \omega\}$ , where  $\bigcup \{A_{\rho \frown n} : n \in \omega\} = X$  and  $|\bigcap \{A_{f \frown n} : n \in \omega\}| \leq 1$  for every  $f \in \omega^\omega$ .

We will show that, for each  $x \in X$ ,  $\{x\} = \bigcap \{St(x, \mathcal{C}_\rho) : \rho \in \omega^{<\omega}\}$ . Suppose that it does not happen. Then there are  $x, y \in X$  with  $x \neq y$  such that for every  $\rho \in \omega^{<\omega}$ ,  $y \in St(x, \mathcal{C}_\rho)$ . Note that, the first move of Player I is  $\{A_n : n \in \omega\}$ . Then player II can choose  $A_{n_0}$  such that  $x, y \in A_{n_0}$ . The next move of Player I is  $\{A_{n_0 n} : n \in \omega\}$  and Player II can choose  $A_{n_0 n_1}$  such that  $x, y \in A_{n_0 n_1}$  and so on. Then there is a  $g \in \omega^\omega$  such that  $x, y \in A_{g \upharpoonright n}$  for each  $n \in \omega$ , so  $|\bigcap_{n \in \omega} A_{g \upharpoonright n}| \geq 2$  which is a contradiction.  $\square$

The Proposition 2.2 is not true for non Lindelöf spaces as we will see in the following. Consider  $\omega_1$  with the usual order topology. Then

**Example 2.3.** Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$  played on  $\omega_1$  and  $S_1(\mathcal{O}, \mathcal{H})$  holds.

*Proof.* Let  $S$  be the set of all successors ordinals less than  $\omega_1$  and  $L = \omega_1 \setminus S$ . For every  $\gamma \in L$  pick a sequence of ordinals  $\{\alpha_n^\gamma : n \in \omega\} \subset \omega_1$  such that  $\alpha_n^\gamma < \alpha_{n+1}^\gamma < \gamma$  and  $\sup\{\alpha_n^\gamma : n \in \omega\} = \gamma$ . For each  $n \in \omega$  let  $V_n^\gamma = ]\alpha_n^\gamma, \gamma]$  and  $A = \{\{\alpha\} : \alpha \in S\}$ .

At the first inning Player I chooses  $\mathcal{C}_0 = A \cup \{V_0^\gamma : \gamma \in L\}$ . Note that if Player II chooses  $\{\alpha\}$  for some  $\alpha \in S$ , then Player II loses the game. Then, we can suppose that Player II chooses  $V_0^{\gamma_0}$  for some  $\gamma_0 \in L$ . At the second inning Player I plays  $\mathcal{C}_1 = A \cup \{]\gamma_0, \omega_1[ \} \cup \{V_1^\gamma : \gamma \in L \text{ and } \gamma \leq \gamma_0\}$ . Suppose Player II chooses  $V_1^{\gamma_1}$  for some  $\gamma_1 \in L$  such that  $\gamma_1 \leq \gamma_0$ . Note that, if  $V_0^{\gamma_0} \cap V_1^{\gamma_1} = \emptyset$  then Player II loses. So, Player I chooses  $\mathcal{C}_2 = A \cup \{]\gamma_1, \omega_1[ \} \cup \{V_2^\gamma : \gamma \in L \text{ and } \gamma \leq \gamma_1\}$  and so on.

Since  $\{\gamma_n : n \in \omega\}$  is a decreasing sequence of ordinals there is a  $k \in \omega$  such that  $\gamma_n = \gamma_k$  for every  $n \geq k$ . So, Player II chooses  $V_n^{\gamma_k} \in \mathcal{C}_n$  for every  $n \geq k$ . Therefore,  $\bigcap_{n \in \omega} V_n^{\gamma_k} \subset \bigcap_{n \geq k} V_n^{\gamma_k} = \{\gamma_k\}$ .

Finally, note that  $\omega_1$  is a countably compact non compact space, therefore,  $\omega_1$  does not have the  $G_\delta$ -property, see e.g. [3]. Thus,  $S_1(\mathcal{O}, \mathcal{H})$  holds.

□

Note that the last proof works for every ordinal with uncountable cofinality. Therefore, the following is true.

**Proposition 2.4.** *Let  $\alpha$  be an ordinal with uncountable cofinality. Then Player I has a winning strategy for  $G_1(\mathcal{O}, \mathcal{H})$  played on  $A_\alpha = \{\beta < \alpha : cf(\beta) = \omega\}$ .*

Now, we will see that the second implication can be reversed for hereditarily metacompact spaces. But, before that, we need some auxiliary results

Let  $Y \subset X$  and let  $\mathcal{O}(Y)$  be the set of all open covers for  $Y$ . Let  $P(X)$  be the following game: At the first inning Player I chooses  $\mathcal{C}_0 \in \mathcal{O}$  and Player II answers by taking  $A_0 \in \mathcal{C}_0$ . At each inning  $n \geq 1$  Player I chooses  $\mathcal{C}_n \in \mathcal{O}(A_{n-1})$  and then Player II answers by taking  $A_n \in \mathcal{C}_n$ . We say that Player II wins if  $|\bigcap_{n \in \omega} A_n| \geq 2$ .

**Proposition 2.5.** *If Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$  then Player I has a winning strategy in  $P(X)$ .*

*Proof.* Let  $\sigma$  be a winning strategy for Player I in  $G_1(\mathcal{O}, \mathcal{H})$ . At the first inning of  $P(X)$  Player I plays  $\sigma(\emptyset)$  and Player II chooses  $A_0 \in \sigma(\emptyset)$ . Then, at the second inning of  $P(X)$ , Player I plays  $\mathcal{C}_1 = \{A_0 \cap A : A \in \sigma(A_0)\}$  and Player II chooses  $C_1 \in \mathcal{C}_1$ . Note that  $C_1 = A_0 \cap A_1$  for some  $A_1 \in \sigma(A_0)$ . So, at the  $n$ th inning of the game  $P(X)$ , Player I plays  $\mathcal{C}_n = \{A_0 \cap A_1 \cap \dots \cap A_{n-1} \cap C : C \in \sigma(A_0, \dots, A_{n-1})\}$  and Player II chooses  $C_n \in \mathcal{C}_n$ , with  $C_n = A_0 \cap A_1 \cap \dots \cap A_n$  for some  $A_n \in \sigma(A_0 \cap A_1 \cap \dots \cap A_{n-1})$ . Therefore  $|\bigcap_{n \in \omega} C_n| = |\bigcap_{n \in \omega} A_n| \leq 1$ , where  $A_0 = C_0$ . □

**Lemma 2.6.** *Let  $X$  be a hereditarily metacompact space. If Player I has a winning strategy in  $P(X)$  then there is a winning strategy for Player I in  $P(X)$  such that Player I only plays point-finite open covers.*

*Proof.* Let  $\sigma$  be a winning strategy for Player I in  $P(X)$ . Let  $\sigma(\emptyset)$  be the first move of Player I and let  $\mathcal{C}_0 = \sigma^*(\emptyset)$  be a point-finite refinement of  $\sigma(\emptyset)$ . If Player II chooses  $A_0^* \in \mathcal{C}_0$ , then there is an  $A_0 \in \sigma(\emptyset)$  such that  $A_0^* \subset A_0$ . Let  $\sigma^*(A_0)$  be a point-finite refinement of  $\sigma(A_0)$ . Let  $\mathcal{C}_1 = \{B^* \cap A_0^* : B^* \in \sigma^*(A_0)\}$  be the play for Player I. Note that  $\mathcal{C}_1$  is a point-finite cover for  $A_0^*$ . Then Player II chooses  $A_1^* \in \mathcal{C}_1$  such that  $A_1^* = B_1^* \cap A_0^*$  with  $B_1^* \in \sigma^*(A_0)$ , then there is an  $A_1 \in \sigma(A_0)$  such that  $B_1^* \subset A_1$ . Let  $\sigma^*(A_0, A_1)$  be a point-finite refinement of  $\sigma(A_0, A_1)$  and Player I plays  $\mathcal{C}_2 = \{B^* \cap A_1^* : B^* \in \sigma^*(A_0, A_1)\}$ . Again, note that  $\mathcal{C}_2$  is a point-finite cover for  $A_1^*$ . Then Player II chooses  $A_2^* \in \mathcal{C}_2$ ,  $A_2^* = B_2^* \cap A_1^*$  with  $B_2^* \in \sigma^*(A_0, A_1)$ , then there is an  $A_2 \in \sigma(A_0, A_1)$  such that  $B_2^* \subset A_2$ .

Proceeding this way, in the  $n$ -th inning Player I plays  $\mathcal{C}_n = \{B^* \cap A_{n-1}^* : B^* \in \sigma^*(A_0, \dots, A_{n-1})\}$  and Player II chooses  $A_n^* \in \mathcal{C}_n$ . Note that for each  $n \in \omega$   $A_n^* \subset A_n$ . Therefore,

$$|\bigcap_{n \in \omega} A_n^*| \leq |\bigcap_{n \in \omega} A_n| \leq 1$$

□

**Proposition 2.7.** *If  $(X, \tau)$  is a hereditarily metacompact space and Player I has a winning strategy in  $P(X)$  then  $S_1(\mathcal{O}, \mathcal{H})$  does not hold.*

*Proof.* As we saw above we can suppose that all covers played by Player I are point-finite. Let  $\sigma$  be a winning strategy for Player I in  $P(X)$ . For each  $x \in X$  let  $S(x, \emptyset) = \{C \in \sigma(\emptyset) : x \in C\}$  and  $S(x, C_0, \dots, C_n) = \{C \in \sigma(C_0, \dots, C_n) : C_n \in S(x, C_0, \dots, C_{n-1}), \dots, C_0 \in S(x, \emptyset) \text{ and } x \in C\}$ . Note that  $S(x, \emptyset)$  and  $S(x, C_0, \dots, C_n)$  are finite sets. For every  $x \in X$ , let  $V_0^x = \bigcap S(x, \emptyset)$  and  $V_n^x = \bigcap \{S(x, C_0, \dots, C_n) : C_n \in S(x, C_0, \dots, C_{n-1}), \dots, C_0 \in S(x, \emptyset)\}$ .

For each  $n \in \omega$  we define  $\mathcal{C}_n = \{V_n^z : z \in X\}$ . We will show that  $\bigcap_{n \in \omega} St(x, \mathcal{C}_n) = \{x\}$ . Suppose that it does not happen, then there are  $x, y \in X$  with  $x \neq y$  such that  $y \in \bigcap_{n \in \omega} St(x, \mathcal{C}_n)$ . Note that if  $St(x, \mathcal{C}_n) = \bigcup \{V_n^z : z \in X \text{ and } x \in V_n^z\}$  then for every  $n \in \omega$  there is  $z_n \in X$  such that  $x, y \in V_n^{z_n}$ . Let  $Lev(n) = \bigcup \{S(x, C_0, \dots, C_n) : C_n \in S(x, C_0, \dots, C_{n-1}), \dots, C_0 \in S(x, \emptyset)\}$  and let  $T = \bigcup_{n \in \omega} Lev(n)$ . Note that  $(T, \leq)$ , is a tree ordered by “ $\supseteq$ ”. Note that  $V_n^{z_n} \in Lev(n)$  for each  $n \in \omega$ .

**Claim.** There is a branch  $R$  of  $(T, \leq)$  such that  $x, y \in \bigcap R$ .

*Proof.* Every level of the tree  $(T, \leq)$  has finitely many elements and each element of a level forks in another finitely many elements of the next level. Then there are  $C_0 \in Lev(0)$  and  $\mathcal{A}_0 \subset T$  with  $|\mathcal{A}_0| = \omega$  such that, for every  $A \in \mathcal{A}_0$ ,  $C_0 \leq A$ . There are  $C_1 \in Lev(1)$  and  $\mathcal{A}_1 \subset \mathcal{A}_0$  with  $|\mathcal{A}_1| = \omega$  such that, for every  $A \in \mathcal{A}_1$ ,  $C_1 \leq A$  and  $C_0 \leq C_1$ . Proceeding this way, we can find for every  $n$  a  $C_n \in Lev(n)$  such that  $C_0 \leq C_1 \leq \dots \leq C_n$  and an  $\mathcal{A}_n \subset \mathcal{A}_{n-1}$  such that  $|\mathcal{A}_n| = \omega$  and, for each  $A \in \mathcal{A}_n$ ,  $C_n \leq A$ . So,  $R = (C_n)_{n \in \omega}$  is a branch and  $x, y \in C_n$  for every  $n \in \omega$ .  $\square$

Therefore, Player II wins, which is a contradiction. Then,  $\bigcap_{n \in \omega} St(x, \mathcal{C}_n) = \{x\}$  for every  $x \in X$ .  $\square$

**Corollary 2.8.** *If  $(X, \tau)$  is a hereditarily metacompact space and Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$  then  $S_1(\mathcal{O}, \mathcal{H})$  does not hold.*

Note that the Pixley-Roy hyperspaces are always hereditarily metacompact, so we have the following corollary:

**Corollary 2.9.** *In any Pixley-Roy hyperspace,  $S_1(\mathcal{O}, \mathcal{H})$  holds if, and only if, Player I does not have a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$ .*

### 3. APPLICATIONS

In [1] it is shown that every space with the countable chain condition and the regular  $G_\delta$ -diagonal property has size at most  $\mathfrak{c}$ . We will show a similar result involving the game  $G_1(\mathcal{O}, \mathcal{H})$ . After that, we will see others applications.

Let  $(X, \tau)$  be a topological space. Let  $\mathcal{S} = \{(S_n)_{n \in \omega} : S_n \in \tau \setminus \{\emptyset\}\}$ . In the following we will denote each sequence  $(S_n)_{n \in \omega}$  only by  $S$ . For each  $A, B \in \mathcal{S}$  with  $A \neq B$  let  $d : \mathcal{S} \times \mathcal{S} \rightarrow [0, 1]$  be a function such that  $d(A, B) = 1/(n+1)$  where  $n = \min\{k \in \omega : A_k \neq B_k\}$  and  $d(A, A) = 0$ . Note that  $(\mathcal{S}, d)$  is a metric space.

Consider  $G_1^*(\mathcal{O}, \mathcal{H})$  the following game: In the  $n$ -th inning, Player I plays a  $C_n \in \mathcal{O}$  and Player II chooses a  $C_n \in \mathcal{C}_n$ . At the end, Player II is the winner if there is a  $k \in \omega$  such that  $|\bigcap_{k \leq n} C_n| \geq 2$ .

Observe that if Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$  then there is a winning strategy for Player I in  $G_1^*(\mathcal{O}, \mathcal{H})$ . Indeed, let  $\sigma$  be a winning strategy for Player I in  $G_1(\mathcal{O}, \mathcal{H})$ . Let us define a strategy for Player I in the  $G_1^*(\mathcal{O}, \mathcal{H})$  in the following way. At the  $n$ -th inning, Player I chooses:

- $\sigma(\emptyset)$ , if  $n = p$  for  $p$  a prime number.
- $\sigma(C_p, \dots, C_{p^{k-1}})$ , if  $n = p^k$  for  $p$  a prime number and for some  $k \in \omega$ ,  $k > 1$ .
- $\sigma(\emptyset)$ , if  $n$  is not a power of a prime number.

Note that, at the  $n$ -th inning, Player II chooses  $C_n$ . So, for any  $k \in \omega$  there is a prime number such that  $p > k$ , then

$$|\bigcap_{n \geq k} C_n| \leq |\bigcap_{n \geq p} C_n| \leq |\bigcap_{n \geq 1} C_{p^n}| \leq 1.$$

**Proposition 3.1.** *Let  $(X, \tau)$  be a metacompact and separable space. If Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$  then  $|X| \leq \mathfrak{c}$ .*

*Proof.* Let  $\sigma$  be the winning strategy for Player I. For each  $x \in X$  let  $S(x, \emptyset) = \{C \in \sigma(\emptyset) : x \in C\}$  and  $S(x, C_0, \dots, C_{n-1}) = \{C_n \in \sigma(C_0, \dots, C_{n-1}) : C_{n-1} \in S(x, C_0, \dots, C_{n-2}), \dots, C_0 \in S(x, \emptyset), \text{ and } x \in C_n\}$ . For each  $x \in X$  let  $\mathcal{S}_x = \{R \in \mathcal{S} : R_n \in \sigma(R_0, \dots, R_{n-1}), \dots, R_0 \in \sigma(\emptyset) \text{ and } \bigcap R = \{x\}\}$ . For each  $x \in X$  and for each  $k \in \omega$  let  $\mathcal{U}_x^k = \{R \in \mathcal{S} : R_n \in S(x, C_0, \dots, C_{n-1}), \dots, C_0 \in S(x, \emptyset) \text{ and } R_n = A_n \text{ for every } n \geq k \text{ and for some } A \in \mathcal{S}_x\}$ . Let  $\mathcal{U}_x = \bigcup_{n \in \omega} \mathcal{U}_x^k$  and let  $\mathcal{U} = \bigcup_{x \in X} \mathcal{U}_x$ . Note that  $(\mathcal{U}, d \upharpoonright \mathcal{U})$  is a metric space and there is a sobrejective function  $g : \mathcal{U} \rightarrow X$  such that  $g(U) = \bigcap U$ .

Let  $D$  be a countable and dense subset of  $X$ . For each  $d \in D$  fix  $A^d \in \mathcal{S}_d$  and for each  $n \in \omega$  let  $\mathcal{D}_d^n = \{H \in \mathcal{U}_d : H_k = A_k^d \text{ for each } k \geq n\}$  and let  $\mathcal{D}_d = \bigcup_{n \in \omega} \mathcal{D}_d^n$ . So, define  $\mathcal{D} = \bigcup_{d \in D} \mathcal{D}_d$ . Observe that  $\mathcal{D}$  is a countable subset of  $\mathcal{U}$ . We will show that  $\overline{\mathcal{D}} = \mathcal{U}$ . Let  $Y \in \mathcal{U}_y$  we will show that  $B(Y, 1/(n+1)) \cap \mathcal{D} \neq \emptyset$ , where  $B(Y, 1/(n+1))$  is the open ball of center  $Y$  and radius  $1/(n+1)$ . For each  $x \in X$  and  $k \in \omega$  let  $V_k^x = \bigcap \{S(x, C_0, \dots, C_{k-1}) : C_{k-1} \in S(x, C_0, \dots, C_{k-1}), \dots, C_0 \in S(x, \emptyset)\}$ . Since  $X$  is a metacompact space,  $\bigcap_{k \leq n} V_k^y$  is a non empty open set then let  $d \in (\bigcap_{k \leq n} V_k^y) \cap D$ . So  $S(d, C_0, \dots, C_k) \supset S(y, C_0, \dots, C_k)$  for each  $k \leq n$ . Let  $H_k = Y_k$  for  $k \leq n$  and let  $H_k = A_k^d$  for  $k > n$ . Therefore  $H \in B(Y, 1/(n+1)) \cap \mathcal{D}$ . Since  $\mathcal{U}$  is a metric space, and a separable then  $\mathcal{U}$  has a countable base. So,  $|\mathcal{U}| \leq \mathfrak{c}$  and since  $g : \mathcal{U} \rightarrow X$  is a sobrejective function the result follows.  $\square$

Note that the metacompactness is important. Consider the Katetov's extension  $K(\omega)$  of  $\omega$ . This is a separable space with  $G_\delta$ -diagonal and have cardinality bigger than  $\mathfrak{c}$ .

**Proposition 3.2.** *Let  $\kappa$  be the Lindelöf degree of a space  $X$ . If Player I has a winning strategy for  $G_1(\mathcal{O}, \mathcal{H})$  then  $|X| \leq \kappa^\omega$ .*

*Proof.* Let  $x \in X$ . At each inning, we can suppose that Player I plays a cover of size  $\kappa$  and Player II chooses an open set which contains the point  $x$ . Then, since Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$ , the intersection of all these open sets is  $\{x\}$ . The size of the set of all branches provided by the winning strategy of Player I is at most  $\kappa^\omega$ .  $\square$

Let  $\kappa$  be an infinite cardinal. Consider the selection principle  $S_1^\kappa(\mathcal{O}, \mathcal{H})$  given by the following statement:

Given  $(\mathcal{C}_\xi)_{\xi < \kappa} \subset \mathcal{O}$ , for each  $\xi < \kappa$  there is a  $C_\xi \in \mathcal{C}_\xi$  such that  $|\bigcap_{\xi < \kappa} C_\xi| \geq 2$ .

Again, we have associated to this principle a game  $G_1^\kappa(\mathcal{O}, \mathcal{H})$  defined as follows.

In the  $\xi$ -th inning, Player I plays a  $C_\xi \in \mathcal{O}$  and Player II chooses a  $C_\xi \in \mathcal{C}_\xi$ . At the end, Player II is the winner if  $|\bigcap_{\xi < \kappa} C_\xi| \geq 2$ .

Note that:

$$\Delta \text{ is } G_\kappa \text{ in } X^2 \Leftrightarrow \neg S_1^\kappa(\mathcal{O}, \mathcal{H}) \Rightarrow \text{I} \uparrow G_1^\kappa(\mathcal{O}, \mathcal{H}).$$

In the following we show that a winning strategy for Player I in  $G_1(\mathcal{O}, \mathcal{H})$  gives a bound for which selections of the form  $S_1^\kappa(\mathcal{O}, \mathcal{H})$  can hold.

**Proposition 3.3.** *Let  $\kappa$  be the Lindelöf degree of a space  $X$ . If Player I has a winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$  then  $S_1^\kappa(\mathcal{O}, \mathcal{H})$  does not hold.*

*Proof.* Without loss of generality, we can assume that at each inning Player I plays an open covering of size  $\kappa$ . Consider a winning strategy for Player I given by the  $\{\mathcal{C}_\rho : \rho \in \kappa^{<\omega}\}$  such that  $\mathcal{C}_\rho = \{A_{\rho \restriction \xi} : \xi < \kappa\}$ ,  $\bigcup \{A_{\rho \restriction \xi} : \xi < \kappa\} = X$  and  $|\bigcap \{A_{f \restriction \xi} : \xi < \kappa\}| \leq 1$  for every  $f \in \kappa^\omega$ .

Note that  $|\kappa^{<\omega}| = \kappa$ . Suppose that  $S_1^\kappa(\mathcal{O}, \mathcal{H})$  holds, i.e., there are  $x, y \in X$  with  $x \neq y$  such that for every  $\rho \in \kappa^{<\omega}$  we have  $y \in St(x, \mathcal{C}_\rho)$ . Then there is a  $g \in \kappa^\omega$  such that  $x, y \in A_{g \restriction \xi}$  for each  $\xi < \kappa$ , so  $|\bigcap_{\xi < \kappa} A_{g \restriction \xi}| \geq 2$ . Therefore, the Player I does not have winning strategy in  $G_1(\mathcal{O}, \mathcal{H})$ .  $\square$

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